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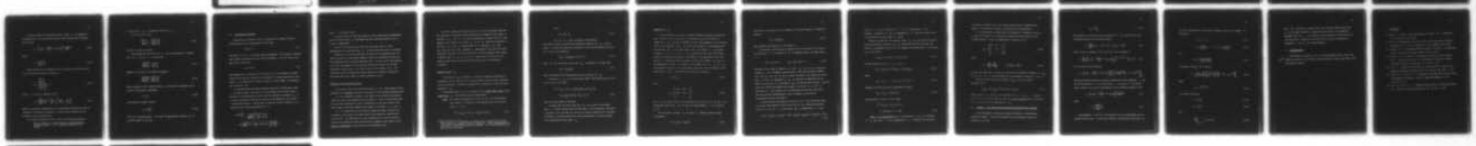

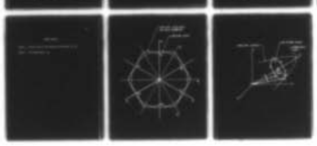

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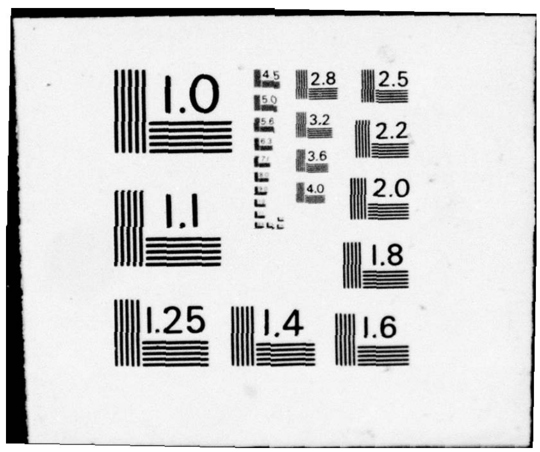
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FRACTURE OF PLASTIC MATERIALS UNDER PROPORTIONAL STRAINING.

Part I: Theoretical Foundations,

~~by~~

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ABSTRACT

A criterion of fracture under proportional straining has been derived for materials that undergo plastic deformation prior to failure. This has been done using endochronic theories of plasticity and fracture proposed by Valanis. The strain criterion for fracture of brittle materials, proposed by Wu, falls into the class of more specific criteria encompassed by the criterion proposed here.

The criterion suggests that fracture will occur when the magnitude s of the strain vector in strain space reaches a critical value. More specifically, in the event that s depends only on the angle ϕ (PHI) between the strain vector and hydrostatic strain vector, it then follows that $s = s(\phi)$, (PHI) thus giving rise to a polar form of a fracture surface which is rotationally symmetric with respect to hydrostatic strain vector. This polar form allows for a <u>one parameter</u> representation of the fracture surface in a <u>triaxial</u> strain field.

In Part II of this paper we show that the theory is fully corroborated by extensive experimental data on gray cast iron.

I. INTRODUCTION

Fracture of materials under proportional loading has been a frequent topic of investigation since it plays an important role in the establishment of safe working loads in engineering design. Numerous fracture criteria have been proposed in the past. A summary of the available theories, though not exhaustive, can be found in Nadai [1] and Paul [2].

In this treatment, we have taken a new approach in obtaining a solution for this old problem. This approach stems from the previous works of the authors.

An energy-probability theory of fracture has recently been proposed by Valanis [3]. This theory is based on the concepts of probability and intrinsic time; the latter was originally introduced by the first author in the development of the endochronic theory of viscoplasticity [4]. According to the aforementioned theory, fracture occurs on an intrinsic time scale ζ which, in the case of plasticity is given by:

$$d\zeta^2 = P_{ijk\ell}\, dE_{k\ell}\, dE_{k\ell} \tag{1.1}$$

where $P_{ijk\ell}$ is a positive semi-definite symmetric material tensor which may depend on the Green tensor E_{ij}. Fracture of a microelement will have occurred if the intrinsic time ζ of the microelement has reached a critical value ζ_c, which is given by the equation:

$$\int_o^{\zeta_c} \left\{1 - e^{-\frac{\gamma}{kT}\psi}\right\} d\zeta = 1 \tag{1.2}$$

where γ is a material parameter; k is the Boltzmann constant; T the absolute temperature; and ψ is the change in energy of the microelement (relative to its unstressed state). The fracture activation energy ψ_o introduced in Ref. [3] has been assumed to be zero in the present discussion.

In the same paper, Valanis discussed the case of fracture under proportional straining when the material obeys Hooke's law. A brief account of this discussion is given in the following:

In the Euclidean principal strain space $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ let:

$$ds^2 = d\varepsilon_i \, d\varepsilon_i \tag{1.3}$$

Proportional straining is then defined by the relation

$$d\varepsilon_i = \ell_i \, ds \tag{1.4}$$

where ℓ_i are constant direction cosines of a radial stain path of length s.

Under conditions of small deformation, the Green tensor E_{ij} in equation (1.1) is replaced by the small strain tensor ε_{ij}. This equation in conjunction with equation (1.4), reduces now to

$$d\zeta^2 = P_{ij} \ell_i \ell_j \, ds^2 \tag{1.5}$$

In Ref. [3] a more restrictive assumption was made, for simplicity, that P_{ij} depends only on the direction of the vector ε_i but not on its magnitude, in which event, the scalar product $P_{ij}\ell_i\ell_j$ is a function of ℓ_i only. In addition, a function $f(\ell_i)$ was defined such that

$$f^2 = P_{ij}\ell_i\ell_j \tag{1.6}$$

Equation (1.5) may therefore be written as

$$d\zeta = f(\ell_i)\, ds \tag{1.7}$$

In the case of elastic materials obeying Hooke's law, the strain energy, which is the free energy at constant temperature, is given by

$$\psi = \alpha s^2 \tag{1.8}$$

where $\alpha = \frac{1}{2} c_{ij}\ell_i\ell_j$ and c_{ij} is the matrix of elastic constants defined by the linear relation $\sigma_i = c_{ij}\varepsilon_j$, where σ_i are the principal components of the stress tensor.

Substituting equation (1.8) into equation (1.2) and integrating, the following transcendental equation is obtained for the critical values s_c of s:

$$f(\ell_i)\left\{ s_c - \sqrt{\frac{\pi kT}{4\gamma\alpha}}\, \mathrm{erf}\sqrt{\frac{\gamma\alpha}{kT}}\, s_c \right\} = 1 \tag{1.9}$$

The solution of this equation is of the form:

$$s_c = s_c(\ell_i) \tag{1.10}$$

<u>Equation (1.10) defines, in strain space, a fracture surface on, or within, which all fracture initiation events will have occurred.</u>

Valanis then considered the case where the fracture surface possesses rotational symmetry with respect to the hydrostatic strain vector (Fig. 1). For this type of fracture surface, s_c is a function only of the angle ϕ

between the strain vector ϵ_i and the hydrostatic strain vector, where

$$\cos \phi = \sqrt{\frac{i_1^2}{3i_2}} = \frac{1}{\sqrt{3}} (\ell_1 + \ell_2 + \ell_3) \tag{1.11}$$

and $i_1 = \mathrm{tr}\underset{\sim}{\epsilon}$, $i_2 = \mathrm{tr}(\underset{\sim}{\epsilon}^2)$.

The function s_c depends on ℓ_i through $\alpha(\ell_i)$ and $f(\ell_i)$. It was shown, however, that any asymmetry of the fracture surface with respect to the null hydrostatic strain plane: $\phi = \frac{\pi}{2}$, must be due to the function $f(\ell_i)$.

In this paper, the same approach as described above with the fracture surface having rotational symmetry about the hydrostatic axis has been followed to investigate fracture of a class of materials which undergo plastic deformation prior to fracture. In particular, a fracture criterion for this class of materials has been derived. It is shown in Part II of this paper, that in the case of gray cast iron, the form of this criterion is identical to the strain criterion of fracture proposed by Wu [5].

The appropriate constitutive equation used in the derivation, is given in the Appendix. This equation becomes linear in s, as s increases beyond the "elastic range", though the magnitude of the strain field remains small in the accepted kinematic sense. The physical consequence in simple tension, for instance, is that the stress-strain curve becomes asymptotically linear, for larger "plastic" strains.

The asymptotic (linear) form of the constitutive equation plays a significant role in the prediction of fracture of plastic materials. The obtained solution is a lower bound.

In the final section, we discuss at length the material tensor P_{ij}

which appears in equation (1.5). This tensor is pivotal in the prediction of fracture after the material has experienced a more general deformation history. It will be shown that, under the assumption of isotropy of the tensor P_{ij} in the null hydrostatic strain plane, the fracture surface is rotationally symmetric about the hydrostatic strain vector. This result will be justified by experimental data on gray cast iron in Part II of this paper.

II. The Free Energy

For plastic materials under proportional straining, an expression for the free energy function ψ is obtained in this section, which, in the next section, is substituted into equation (1.2) to obtain the fracture criterion.

In the case of small deformation and under suitable smoothness assumptions regarding the functional dependence of ψ on $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{q}^{\alpha}$, one can express the free energy density in a quadratic form in $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{q}^{\alpha}$, where $\underset{\sim}{q}^{\alpha}$ $(\alpha = 1, 2, \ldots, n)$ are n internal state variables. In an one dimensional stress field, a direct comparison with a mechanical model shows that the appropriate form is:

$$\psi = \frac{1}{2} \sum_{\alpha} E_{\alpha}(\varepsilon_1 - q_1^{\alpha})^2 \tag{2.1}$$

where E_{α} are the spring constants and ε_1 and q_1^{α} are, respectively, the strain and internal state variables corresponding to the one-dimensional field.

In a three-dimensional situation, isotropy allows a simple decomposition of ψ into the form:

$$\psi = \psi_H + \psi_D \tag{2.2}$$

where ψ_H and ψ_D are the parts of free energy associated with volumetric deformation and shear deformation respectively, as can be readily seen in Appendix II of Ref. [4]. We thus have:

$$\psi_D = \sum_\alpha \mu_\alpha \mathrm{tr}\ [(\underset{\sim}{e} - \underset{\sim}{p}^\alpha)^2] \tag{2.3}$$

and

$$\psi_H = \frac{1}{2} \sum_\alpha K_\alpha\ (\varepsilon - q^\alpha)^2 \tag{2.4}$$

where

$$\underset{\sim}{p}^\alpha = \underset{\sim}{q}^\alpha - \frac{1}{3}\underset{\sim}{\delta}\, q^\alpha$$

$$q^\alpha = \mathrm{tr}\ \underset{\sim}{q}^\alpha$$

$$\underset{\sim}{e} = \underset{\sim}{\varepsilon} - \frac{1}{3}\underset{\sim}{\delta}\, \varepsilon$$

$$\varepsilon = \mathrm{tr}\ \underset{\sim}{\varepsilon} \tag{2.5}$$

The constants μ_α and K_α are the spring constants corresponding to shear and volumetric deformations, respectively.

Using equations (2.2 - 2.5) and following Ref. [4], the rate equations of the internal state variables are obtained in the principal strain space in the following form:

$$K_\alpha q^\alpha + b_o^\alpha \frac{dq^\alpha}{dz} = K_\alpha\ \varepsilon \quad , \qquad (\alpha \text{ not summed}) \tag{2.6}$$

$$2\mu_\alpha p_i^\alpha + b_2^\alpha \frac{dp_i^\alpha}{dz} = 2\mu_\alpha\ e_i \qquad (\alpha \text{ not summed}) \tag{2.7}$$

where b_o^α and b_2^α are the volumetric and the deviatoric parts of the viscosity tensor respectively and z is the intrinsic time scale.

Integration of equations (2.6) and (2.7) gives:

$$q^\alpha = \varepsilon(z) - \int_o^z e^{-\lambda_\alpha (z-z')} \frac{\partial\varepsilon}{\partial z'}\, dz' \tag{2.8}$$

and

$$p_i^\alpha = e_i(z) - \int_o^z e^{-\lambda_\alpha (z-z')} \frac{\partial e_i}{\partial z'}\, dz' \tag{2.9}$$

where, for constant Poisson's ratio*,

$$\lambda_\alpha = \frac{K_\alpha}{b_o^\alpha} = \frac{2\mu_\alpha}{b_2^\alpha} \tag{2.10}$$

One can combine equations (2.8) and (2.9) to obtain:

$$q_i^\alpha = \varepsilon_i - \int_o^z e^{-\lambda_\alpha (z-z')} \frac{\partial\varepsilon_i}{\partial z'}\, dz' \tag{2.11}$$

We now observe that since for proportional straining $\varepsilon_i = \ell_i s$ and $\varepsilon = \sqrt{3}\ s\cos\phi$, equation (2.11) can be reduced to:

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* This is a reasonable assumption as has been demonstrated in Ref.'s [6] and [7].

$$q_i^\alpha = \ell_i \bar{q}^\alpha \tag{2.12}$$

where

$$\bar{q}^\alpha = \frac{q^\alpha}{\sqrt{3}\cos\phi} \tag{2.13}$$

<u>Equation (2.12) shows that for proportional straining, the internal variables and the strain vector are collinear in strain space - if Poisson's ratio is a constant function.</u>

Equations (2.2), (2.3), (2.4), (2.8) and (2.9) lead to the expression:

$$\psi = \frac{1}{2}\left[3K_o\cos^2\phi + 2\mu_o\sum_i\left(\ell_i - \frac{\cos\phi}{\sqrt{3}}\right)\left(\ell_i - \frac{\cos\phi}{\sqrt{3}}\right)\right]$$

$$\cdot \sum_\alpha G_\alpha\left[\int_o^z e^{-\lambda_\alpha(z-z')}\frac{\partial s}{\partial z'}\,dz'\right]^2 \tag{2.14}$$

where, for constant Posson's ratio, the heredity functions are given by:

$$\mu(z) = \mu_o G(z) \quad , \qquad K(z) = K_o G(z) \tag{2.15a}$$

where

$$G(0) = 1 \tag{2.15b}$$

$$G(z) = \sum_\alpha G_\alpha e^{-\lambda_\alpha z} \quad , \qquad \sum_\alpha G_\alpha = 1 \tag{2.15c}$$

and

$$\mu_\alpha = \mu_o G_\alpha \quad , \qquad K_\alpha = K_o G_\alpha \tag{2.15d}$$

The integral in equation (2.14) can now be expressed in closed form by use of equation (1.7) and the following relation previously introduced by Valanis in Ref. [4],

$$z = \frac{1}{\beta} \log (1 + \beta\zeta) \tag{2.16}$$

where β is a constant. The resulting expression is:

$$\psi = \frac{1}{2}\left(3K_o \cos^2\phi + 2\mu_o \sin^2\phi\right) \cdot \sum_\alpha G_\alpha \left\{\left(\frac{1 + \beta fs}{\beta f n_\alpha}\right)\left[1 - \left(1 + \beta fs\right)^{-n_\alpha}\right]\right\}^2 \tag{2.17}$$

where

$$n_\alpha = 1 + \frac{\lambda_\alpha}{\beta} \tag{2.18}$$

Equation (2.17) represents the free energy for plastic materials under proportional straining. For large s, the free energy ψ has the asymptotic representation ψ_A, where

$$\psi_A = \frac{1}{2}\left(3K_o \cos^2\phi + 2\mu_o \sin^2\phi\right) \cdot \sum_\alpha G_\alpha \left(\frac{1 + \beta fs}{\beta f n_\alpha}\right)^2 \tag{2.19}$$

or

$$\psi_A = \frac{1}{2}\left[3K_o \cos^2\phi + 2\mu_o \sin^2\phi\right]\left[\frac{1 + \beta fs}{\beta f \bar{n}}\right]^2 \tag{2.20}$$

where

$$\frac{1}{\bar{n}^2} = \sum_\alpha \frac{G_\alpha}{n_\alpha^2} \tag{2.21}$$

It will be shown in the Appendix that for large s, the asymptotic representation of the constitutive equation under proportional straining has the form:

$$\sigma_i = \left[2\mu_o\left(\ell_i - \frac{\cos\phi}{\sqrt{3}}\right) + \sqrt{3}\, K_o \cos\phi\right] \frac{1 + \beta f s}{\beta f n} \tag{2.22}$$

where

$$n = 1 \Bigg/ \sum_\alpha \frac{G_\alpha}{n_\alpha} \tag{2.23}$$

Direct substitution of equation (2.22) in equation (2.19) gives rise to the following expression for large s:

$$\psi_A = \frac{\sum_\alpha \frac{G_\alpha}{n_\alpha^2}}{\left[\sum_\alpha \frac{G_\alpha}{n_\alpha}\right]^2} \psi_\sigma \tag{2.24}$$

where ψ_σ is given by equation (2.25):

$$\psi_\sigma = \frac{1}{2}\left[\sum_i \frac{\sigma_i \sigma_i}{2\mu_o} - \left[\sum_i \sigma_i\right]^2 \frac{1}{3}\left(\frac{1}{2\mu_o} - \frac{1}{3K_o}\right)\right] \tag{2.25}$$

and has the physical significance of the recoverable energy which would be obtained if the material unloaded in a linear elastic fashion from the stressed to the unstressed state.

As a result of equation (2.24) we have the following consequence:

> The free energy of a plastic material under proportional loading and large s, is a quadratic function of the stress.

In other words, ψ_A is a quadratic function of σ_i.

Also one can show that:

$$\sum \frac{G_\alpha}{n_\alpha^2} \geq \left[\sum \frac{G_\alpha}{n_\alpha}\right]^2 \tag{2.26}$$

This may be shown as follows:

From equation (2.15c) and since G_α are all non-negative it follows that $G_\alpha \leq 1$ for all α. In that event,

$$\sum_\alpha \left[\frac{G_\alpha}{n_\alpha}\right]^2 \leq \sum_\alpha \frac{G_\alpha}{n_\alpha^2} \tag{2.27}$$

However, as a result of the Bessel Inequality:

$$\sum \left[\frac{G_\alpha}{n_\alpha}\right]^2 \geq \left[\sum \frac{G_\alpha}{n_\alpha}\right]^2 \tag{2.28}$$

Hence, equation (2.26) follows directly. As a result of inequality (2.26) we have the following consequence:

$$\psi_A \geq \psi_\sigma \tag{2.29}$$

For instance in simple tension:

$$\psi_A \geq \frac{1}{2} \frac{\sigma^2}{E_o}. \tag{2.30}$$

which is a revealing result. Note that for <u>one</u> internal variable ψ_A is precisely equal to $\frac{1}{2}\sigma^2/E_o$.

III. The Fracture Criterion

Substitution of equation (2.17) in equation (1.2) yields a fracture criterion which has a representation of the type:

$$F(\phi, s_c) = 1 \tag{3.1}$$

from which s_c of s at fracture may be determined. This fracture criterion has a number of features which are worth discussing. It is of a polar form which can be written as:

$$s_c = F^*(\phi) \tag{3.2}$$

The significance of equation (3.2) is that it is an one parameter representation of the fracture surface in the presence of a three dimensional strain field. It is analytically defined in terms of a small number of parameters, and a function $f(\phi)$.

In the particular case where fracture occurs for a sufficiently large s (in simple tension this would be on the straight part of the "plastic" portion of the stress-strain curve), then the asymptotic expression ψ_A of ψ given by equation (2.20) may be used as an approximation; this leads to a closed form solution. Substitution of equation (2.20) in equation (1.2), then gives the following expression for s_c.

$$f(\cos\phi)\left\{s_c - \frac{\bar{n}}{2\sqrt{\frac{\gamma\mu_o}{\pi kT}\left(1 + \frac{3\nu}{1-2\nu}\cos^2\phi\right)}}\right.$$

$$\left.\cdot\ \operatorname{erf}\left[\sqrt{\frac{\gamma\mu_o}{kT}\left(1 + \frac{3\nu}{1-2\nu}\cos^2\phi\right)}\cdot\left(\frac{1 + \beta f s_c}{\beta f \bar{n}}\right)\right]\right\} = 1 \tag{3.3}$$

where ν is Poisson's ratio.

It may be noted that the free energy is, thus, continuously overestimated and the critical value s_c determined from the resulting equation (3.3) is, in fact a LOWER BOUND.

In Part II of this paper we show that the lower bound is a good approximation to the actual value s_c, as far as gray cast iron is concerned.

Equation (3.3) has been derived on the basis of the assumption that the fracture surface is rotationally symmetric about the hydrostatic strain axis. This assumption will be further discussed later. We also notice from equation (3.3) that any asymmetry of the fracture surface with respect to the null hydrostatic strain plane $\phi = \frac{\pi}{2}$ must be due to the function $f(\ell_1)$. This conclusion has been obtained for elastic materials by Valanis [3] and has been shown here to be valid for plastic materials as well.

Symmetries of the Fracture Surface

The fracture locus in the deviatoric plane $(\phi = \frac{\pi}{2})$ must possess certain symmetries for isotropic materials on the basis of the following considerations:

(1) With reference to Figure 1, the shape of the fracture locus should remain invariant with permutation of the indices 1, 2 and 3 of the strain-axes. The constraint resulting from this consideration is that the fracture locus must possess a three-fold symmetry with respect to the axes ε_1, ε_2 and ε_3.

(2) It is observed that the fracture properties remain invariant with a reversal of the sign of the shear field in the presence of uniform hydrostatic strain. The implication of this consideration is that the fracture locus in the deviatoric strain plane should be symmetric with respect to the point of intersection of this plane and the hydrostatic axis.

The first consideration requires that, in so far as isotropic materials are concerned, a proposed fracture surface must be symmetric with respect to the axes ε_1, ε_2 and ε_3; the second consideration requires that it is also symmetric with respect to the axes ε_1', ε_2' and ε_3' which are orthogonal to the axes ε_1, ε_2 and ε_3 respectively, at 0. Therefore, $OA_1 = OA_1'$, $OA_2 = OA_2'$ and $OA_3 = OA_3'$ in Figure 1. Combination of the above two considerations* requires of the fracture surface a six-fold symmetry with respect to the six axes ε_1, ε_2, ε_3; ε_1', ε_2', ε_3'. Thus, only a 30° sector of the fracture locus need to be found for the entire locus to be completely determined.

<u>Material Tensor</u> P_{ij}

If the matrix P_{ij} were known, f would be completely determined in consequence of the relation (1.6), and the fracture surface would, therefore, also be determined. Conversely, equation (3.1) contributes toward the determination of P_{ij}.

The material matrix P_{ij} is now shown to be <u>a second order tensor in the tensor space</u> ε_i. This follows from the axiom below:

<u>Axiom</u>: The functional dependence of $d\zeta^2$ on $d\varepsilon_i$ is form-invariant with respect to rotation of the coordinate axes in the tensor space ε_i, i.e.,

$$d\zeta^2 = P_{ij}(\varepsilon_k)\, d\varepsilon_i\, d\varepsilon_j = P'_{ij}(\varepsilon'_k)\, d\varepsilon'_i\, d\varepsilon'_j \tag{3.4}$$

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* These arguments are reminiscent of similar ones in conjunction with the yield locus in the classical theory of plasticity. There, incompressibility and absence of Bauschinger effect were assumed. Such considerations do not arise in this discussion.

where

$$d\varepsilon'_i = Q_{ij}\ d\varepsilon_j \tag{3.5}$$

and Q_{ij} is a proper orthogonal transformation.

This axiom is then a necessary and sufficient condition for P_{ij} to be a tensor. To prove the sufficient condition, we note that equations (3.4) and (3.5) lead to the expression

$$(P_{k\ell} - P'_{ij}Q_{ik}Q_{j\ell})\ d\varepsilon_k\ d\varepsilon_\ell = 0 \tag{3.6}$$

Since $d\varepsilon_i$ is an arbitrary vector and P_{ij} is symmetric it follows that

$$P_{k\ell} = Q_{ik}Q_{j\ell}P'_{ij} \tag{3.6}$$

which establishes the aforementioned tensorial property of P_{ij}.

Conversely, if P_{ij} is a second order tensor, then substitution of (3.5) and (3.7) into (3.4) gives

$$\begin{aligned} d\zeta^2 &= P_{ij}\ d\varepsilon_i\ d\varepsilon_j = Q_{ki}Q_{\ell j}P'_{k\ell}Q_{ir}\ d\varepsilon'_r\ Q_{js}\ d\varepsilon'_s \\ &= \delta_{kr}\delta_{\ell s}P'_{k\ell}\ d\varepsilon'_r\ d\varepsilon'_s = P'_{k\ell}\ d\varepsilon'_k\ d\varepsilon'_\ell \end{aligned} \tag{3.8}$$

Thus, the above axiom is recovered.

We remark that the above proof that P_{ij} is a tensor in the tensor space ε_i follows directly from the definition of the endochronic time given by equation (1.1), and does not involve the fracture properties of a material. Furthermore, the above proof may be easily generalized to include the case of six-dimensional tensor space ε_{ij}.

Symmetries of P_{ij}

It is convenient to introduce a primed orthogonal coordinate system such that 3'-axis is also the axis of the hydrostatic strain (see Figure 2). Symmetries of P_{ij} can now be deduced from the 6-fold symmetry of the fracture locus in the deviatoric plane. The latter also implies a 6-fold symmetry for the function $f(\ell_i)$ on the same plane. Thus, the value of f is the same at points A, B, C, D, E and F. In the particular case, when the fracture surface is rotationally symmetric with respect to the hydrostatic axis, the value of f remains the same everywhere on the circle ABCDEF. Let P'_{ij} be the components of tensor $\underset{\sim}{P}$ referred to the primed axes; let ℓ'_i and ℓ''_i be the direction cosines of two proportional strain paths that correspond to the same value of $f(\ell'_i)$. These direction cosines are related by

$$\ell''_i = R_{ij}\ell'_j \tag{3.9}$$

where

$$R_{ij} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.10}$$

and θ is the angle between the projections of the vectors ℓ'_i and ℓ''_i in the deviatoric strain plane. For the case of 6-fold symmetry, θ is a multiple of $\frac{\pi}{3}$.

For the class of vectors ℓ''_i for which f^2 remains constant we have the relation

$$f^2 = P'_{ij}\ell'_i\ell'_j = P'_{ij}\ell''_i\ell''_j \tag{3.11}$$

Substitution of equation (3.9) in equation (3.11) yields readily the important relation

$$P'_{ij} = R_{ik}R_{j\ell}P'_{k\ell} \tag{3.12}$$

which defines the symmetries of the tensor $\underset{\sim}{P}$.

In particular, substitution of equation (3.10) in equation (3.12) and in the event that θ is a multiple of $\frac{\pi}{3}$, it transpires that, for these values of θ:

$$P'_{11} = P'_{22} \neq P'_{33} \quad , \qquad P'_{12} = P'_{23} = P'_{31} = 0 \tag{3.13}$$

Therefore, at the axes of symmetry, the tensor P_{ij} has only two independent components. The off-diagonal terms are zero. This does not imply, however, that P'_{12}, P'_{23} and P'_{13} are zero at other locations in the deviatoric strain plane. In fact, all components of P'_{ij} are periodic in magnitude with an angle of periodicity $\frac{\pi}{3}$. Furthermore, P'_{ij} is positive semi definite. P'_{ij} also enjoys the reflective symmetry with respect to the axes ε'_1, ε'_2 and ε'_3 (see Figure 1). This implies that in determining the fracture locus of an isotropic material experimentally, only a 30 degree sector need be investigated as stated previously.

In the particular case where the fracture locus in the deviatoric plane is circular then equation (3.12) is true for all R_{ij} defined by equation (3.10), i.e., for all θ. In this event P'_{ij} is independent of θ and

$$f(\ell_i)^2 = P'_{ij}\ell'_i\ell'_j = P'_{11}(\ell'^2_1 + \ell'^2_2) + P'_{33}\ell'^2_3 = P'_{11}\sin^2\phi + P'_{33}\cos^2\phi = f^2(\phi) \tag{3.14}$$

That is, f is only a function of ϕ. If ϕ is fixed, then f is a constant. Conversely, if P'_{ij} is independent of θ, then the fracture locus in the deviatoric strain plane is a circle.

Finally, we show that the material tensor P_{ij} can not be fully isotropic. This can be shown by assuming P_{ij} to be isotropic and then show that this leads to a consequence which is not in agreement with observation. For isotropic P_{ij}, we can write

$$P_{ij}(\varepsilon_k) = P_{ij}(\varepsilon_k \varepsilon_\ell) = P_{ij}(s, \ell_k \ell_\ell) \tag{3.15}$$

The most general isotropic of P_{ij} form is

$$P_{ij} = P_o(s)\delta_{ij} + P_1(s)L_{ij} + P_2(s)L_{ik}L_{kj} \tag{3.16}$$

where

$$L_{ij} = \ell_i \ell_j \quad , \qquad \text{tr } \underset{\sim}{L} = \text{tr } \underset{\sim}{L}^2 = \text{tr } \underset{\sim}{L}^3 = 1 \tag{3.17}$$

Equation (3.16) can further be simplified to obtain

$$P_{ij} = P_o \delta_{ij} + (P_1 + P_2)\ell_i \ell_j \tag{3.18}$$

Substitution of (3.18) in (1.6) yields

$$f^2 = P_{ij}\ell_i \ell_j = P_o + P_1 + P_2$$

$$= \text{a function of } s \text{ only} \tag{3.19}$$

<u>Thus, f is independent of ϕ</u>. In particular, if P_{ij} is a function of ℓ_1 only, then f is also independent of s. Therefore, as the result

of fracture criterion (3.3), the fracture surface must be symmetric with respect to the deviatoric plane in contradiction with observation.

It is interesting to remark that the assumption of transverse isotropy of P'_{ij} with respect to the hydrostatic strain axis leads readily to the following result, though the application of equation (3.6):

$$P_{ij} = N \begin{bmatrix} 1-M & M & M \\ M & 1-M & M \\ M & M & 1-M \end{bmatrix} \tag{3.20}$$

where

$$M = -\frac{\left(P'_1 - P'_3\right)}{\left(P'_1 + 2P'_3\right)} \quad , \qquad N = \frac{1}{3}\left(P'_1 + 2P'_3\right) \, , \tag{3.21}$$

In the above equations, P'_1 and P'_3 are the two independent parameters in P'_{ij}. We note that P_{ij} is thus determined by two independent parameters M and N.

This last form (3.20) also follows if $P_{ijk\ell}$ is of the special isotropic form:

$$P_{ijk\ell} = P_1 \delta_{ij} \delta_{k\ell} + \frac{1}{2} P_2 \; \delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk} \tag{3.22}$$

where, however, P_1 and P_2 are functions of $\cos\phi$ which is given by eq. (1.11); in other words P_1 and P_2 <u>are not constants</u> but are functions of a certain sort of the invariants of ε_{ij}.

IV. APPENDIX - THE STRESS-STRAIN RELATION UNDER PROPORTIONAL STRAINING

In the discussion of fracture of plastic materials, a stress-strain relation is useful. This can be derived from the endochronic theory of plasticity, in which

$$\sigma_{ij} = \frac{\partial\psi}{\partial\varepsilon_{ij}} \tag{A.1}$$

This equation, in conjuction with equations (2.2 - 2.4), gives rise to the following equation in the principal space:

$$\sigma_i = \sum_\alpha \left[2\mu_\alpha(\varepsilon_i - q_i^\alpha) + (K_\alpha - \frac{2}{3}\mu_o)(\varepsilon - q^\alpha)\right] \tag{A.2}$$

which, by use of equations (2.8) and (2.11), can be reduced to

$$\sigma_i = \sum_\alpha \left\{\left[2\mu_\alpha\left(\ell_i - \frac{\cos\phi}{\sqrt{3}}\right) + \sqrt{3}\, K_\alpha \cos\phi\right] \int_o^z e^{-\lambda_\alpha(z-z')} \frac{\partial s}{\partial z'}, dz'\right\} \tag{A.3}$$

or, for proportional straining:

$$\sigma_i = \left[2\mu_o\left(\ell_i - \frac{\cos\phi}{\sqrt{3}}\right) + \sqrt{3}\, K_o \cos\phi\right] \sum_\alpha \left\{G_\alpha\left(\frac{1+\beta fs}{\beta fn_\alpha}\right)\left[1 - \left(1+\beta fs\right)^{-n_\alpha}\right]\right\} \tag{A.4}$$

The above equation governs the relation between the stress components σ_i and the strain magnitude s for a given angle ϕ. For large s, the stress-strain relation (A.4) may be approximated by the following asymptotic form:

$$\sigma_i = \left[2\mu_o\left(\ell_i - \frac{\cos\phi}{\sqrt{3}}\right) + \sqrt{3}\, K_o \cos\phi\right]\left(\frac{1+\beta fs}{\beta fn}\right) \tag{A.5}$$

where

$$n = 1 \Big/ \sum_\alpha \frac{G_\alpha}{n_\alpha} \tag{A.6}$$

The parameters n and βf in equation (A.5) can be determined from the uniaxial tension test. In this test, one has a proportional stress path; one

also has a proportional strain path if Poisson's ratio is a constant. In particular,

$$\varepsilon_2 = \varepsilon_3 = -\nu\ \varepsilon_1 \tag{A.7}$$

$$\ell_1 = \frac{1}{\sqrt{1 + 2\nu^2}}\ , \qquad \ell_2 = \ell_3 = \frac{-\nu}{\sqrt{1 + 2\nu^2}} \tag{A.8}$$

and

$$\cos\phi = \frac{1}{\sqrt{3}}\left(\frac{1 - 2\nu}{\sqrt{1 + 2\nu^2}}\right) \tag{A.9}$$

Furthermore, equation (A.4) becomes

$$\sigma_1 = \frac{E_o}{\sqrt{1 + 2\nu^2}} \sum_{\alpha} \left\{ G_\alpha \left(\frac{1 + \beta_1\varepsilon_1}{\beta f n}\right)\left[1 - \left(1 + \beta_1\varepsilon_1\right)^{-n_\alpha}\right]\right\} \tag{A.10}$$

where

$$\beta_1 = \sqrt{1 + 2\nu^2}\ \beta f \tag{A.11}$$

It is easily shown that

$$n = E_o/E_t \tag{A.12}$$

$$\beta_1 = E_t/\sigma_o \tag{A.13}$$

and

$$\left.\frac{d\sigma_1}{d\varepsilon_1}\right|_{\varepsilon_1 \to 0} = E_o \equiv E(o) \tag{A.14}$$

where $E(o)$ denotes the initial slope of the uniaxial stress-strain curve, E_t is the tangent modulus at large strain (the slope of the asymptotic straight line), and σ_o the intercept of the asymptotic straight line with the stress axis. Equations (A.12) and (A.13) are used to determine the parameters n and β_1 for a given material.

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NEW

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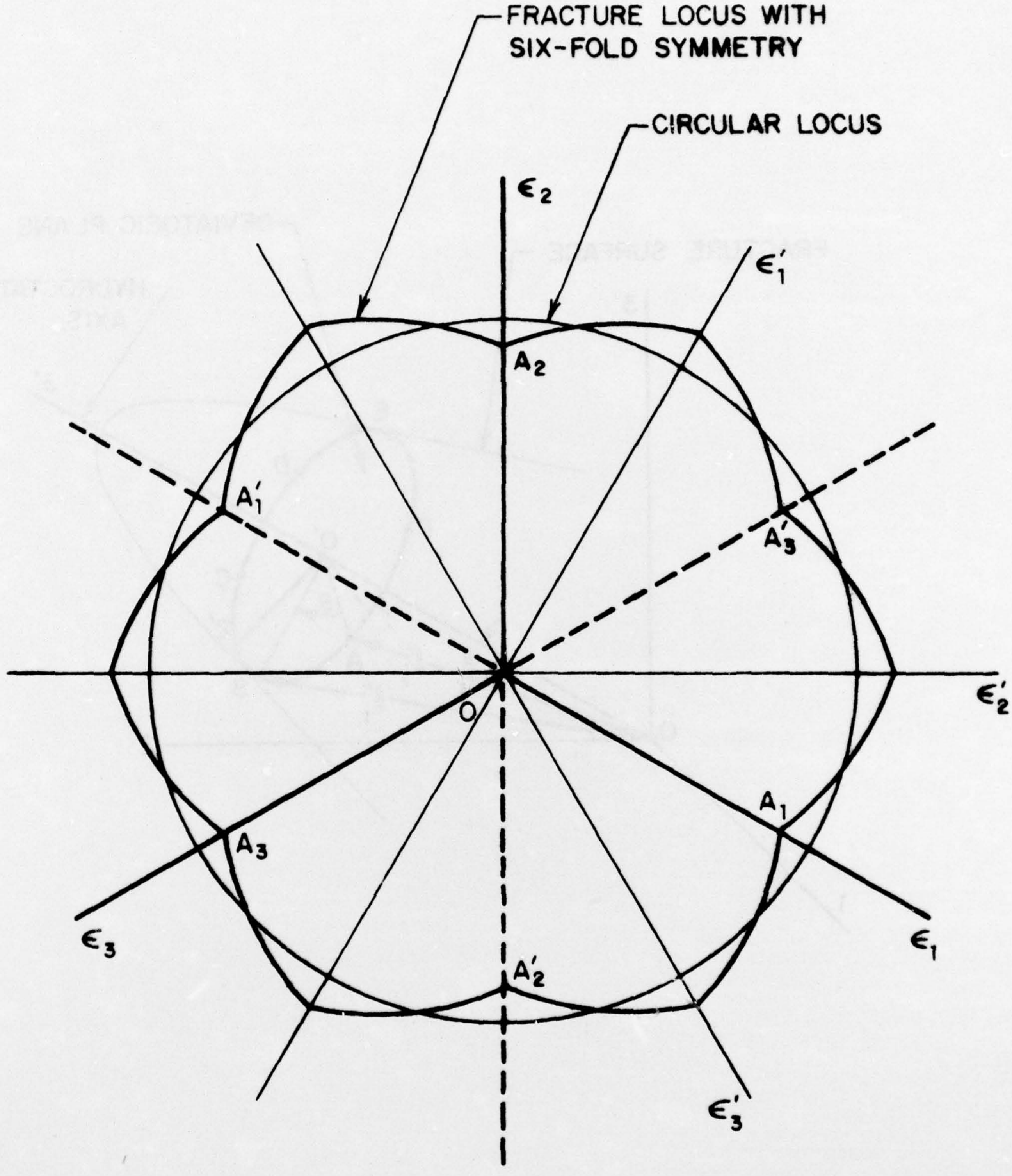
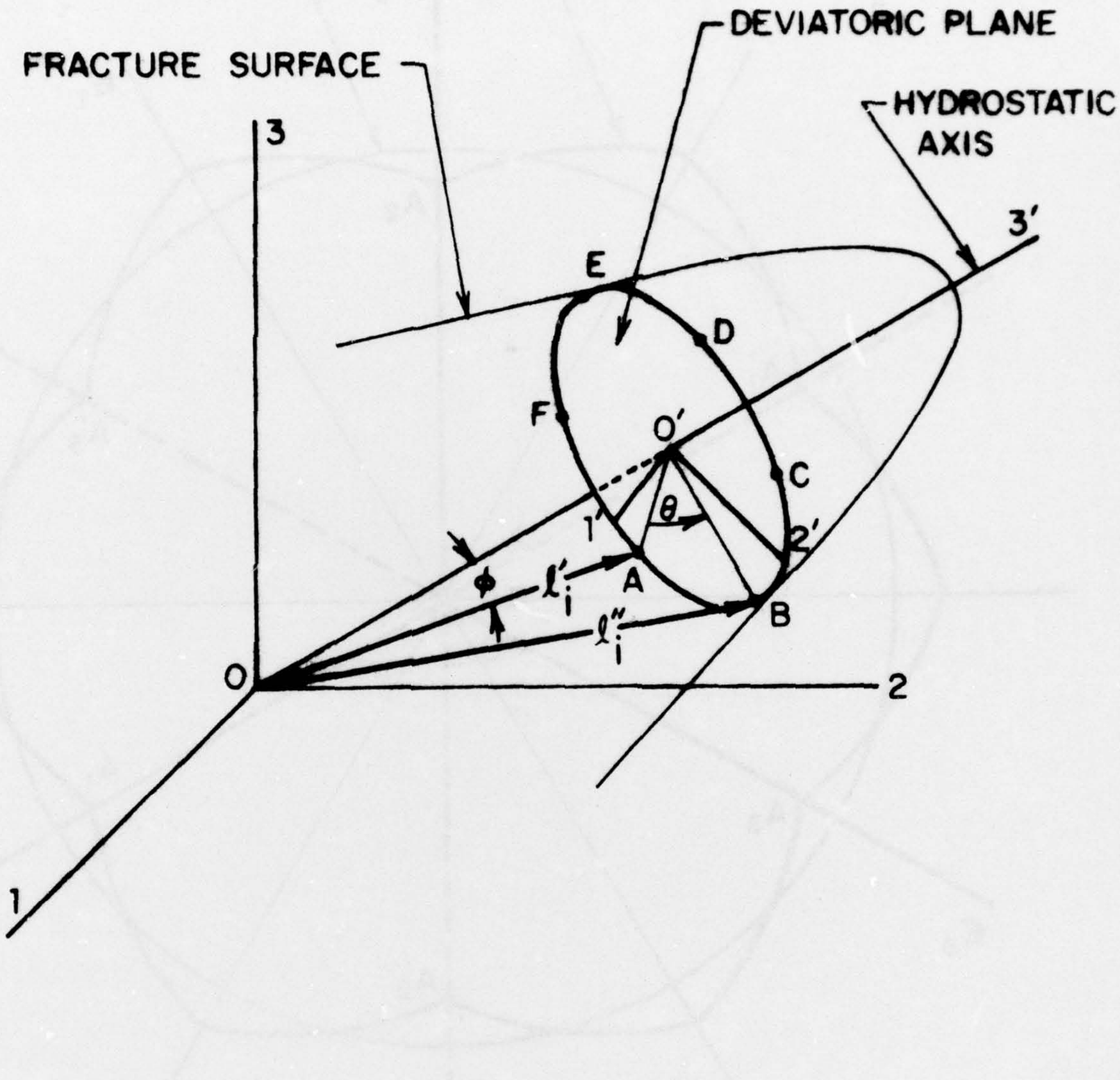
FIGURE CAPTIONS

Figure 1 Fracture locus in the deviatoric strain plane ($\phi = \frac{\pi}{2}$)

Figure 2 The transformation R_{ij}


FRACTURE LOCUS WITH SIX-FOLD SYMMETRY
CIRCULAR LOCUS
ϵ_2
ϵ_1'
A_2
A_1'
A_3'
ϵ_2'
O
A_1
A_3
ϵ_3
ϵ_1
A_2'
ϵ_3'



DEVIATORIC PLANE
FRACTURE SURFACE
HYDROSTATIC AXIS
3
3′
E
D
F
O′
C
1′
θ
2′
ϕ
l_i'
A
B
l_i''
O
2
1